

Exactly solvable hydrogen-like potentials and factorization method

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Abstract

A set of factorization energies is introduced, giving rise to a generalization of the Schrödinger (or Infeld and Hull) factorization for the radial hydrogen-like Hamiltonian. An algebraic intertwining technique involving such factorization energies leads to derive n -parametric families of potentials in general almost-isospectral to the hydrogen-like radial Hamiltonians. The construction of SUSY partner Hamiltonians with ground state energies greater than the corresponding ground state energy of the initial Hamiltonian is also explicitly performed.

Key-Words: Factorization, Hydrogen atom

PACS: 03.65.Ge, 03.65.Fd, 03.65.Ca

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1 Introduction

The factorization method has been historically attributed to Schrödinger [1]. This approach (originated from the well known treatment of the harmonic oscillator in non-relativistic Quantum Mechanics) avoids the use of cumbersome mathematical tools. Since Schrödinger the factorization method has been successfully applied to solve essentially any problem for which there exists exact solution [2]. Considering the narrow set of exactly solvable potentials, it is interesting to look for new potentials for which the corresponding Schrödinger equation becomes analytically solvable. There are a lot of results on the matter, in particular it is worthwhile mentioning a work of Mielnik [3], where a variant of the standard factorization was introduced in order to get a family of Hamiltonians isospectral to the harmonic oscillator. The first application of Mielnik's factorization was performed for the hydrogen-like radial potentials [4], the resulting family of Hamiltonians became isospectral to the standard radial hydrogen-like Hamiltonians. An additional interesting application arose from the arena of the coherent states. Concerning this subject, it is well known that the usual first order factorization operators for the harmonic oscillator Hamiltonian are also its creation and annihilation operators. This is not the case for the isospectral oscillator Hamiltonians derived by Mielnik because their first order factorization operators are different from their creation and annihilation operators, which turn out to be of order equal to (or greater than) three [3, 5]. These last operators are the generators of non-linear algebras, and allow one to get immediately the corresponding coherent states as the eigenstates of the annihilation operator [5]. Recently, these algebras have been rederived (using different methods) and reinterpreted by other authors [6, 7]. It has been shown as well that such non-linear algebras can be associated to other systems [7].

The factorization techniques [1, 2, 3], the Darboux transformation [8], the supersymmetric Quantum Mechanics (SUSY QM) [9], and other procedures have a common point: they can be embraced in the algebraic scheme known as *first order intertwining transformation* which makes use of a first order differential operator to *intertwine* (the eigenfunctions of) two different Hamiltonians [10]. The generalization of this technique involves higher order differential operators and leads in a natural way to the higher order SUSY QM [11-15]. One of the advantages of the intertwining technique is that the composition of n first order intertwining transformations allows one to express the n th-order intertwiner operators as the product of n first order intertwining operators [11, 13, 15].

The experimental study of spectra (for systems as nuclei, atoms, molecules, etc) makes important *per se* the theoretical determination of the energy levels of potentials capable of being used as models for describing different physical situations. However, for a quantum system with a given Hamiltonian it is seldom possible to get exact solutions to the corresponding eigenproblem.

In this paper we are going to show that, given an exactly solvable potential as an input, the *intertwining technique* provides, in general, new exactly solvable potentials which, from the traditional viewpoint, would be solved by perturbative or approximate methods. Hence, working with Hamiltonians whose spectrum is known, one has at hand the tools to test the validity and convergence of some perturbative or approximate methods [13, 16]. In Section 2 we shall present the n th-order intertwining technique, discussing

in more detail those aspects which have not been previously reported in the literature. In particular, in Section 2.1 we shall show that the standard Infeld-Hull factorization as well as the modified Mielnik's method can be recovered by the first order intertwining treatment. We will introduce a generalization of the Mielnik's approach by using solutions to the Schrödinger equation corresponding to factorization energies not belonging to the physical spectrum of the initial problem. In Section 2.2, by using the second order intertwining approach, we will introduce explicit expressions allowing the construction of *two*-parametric families of potentials whose spectrum is equal to the initial spectrum plus two new levels at predetermined positions. In Section 2.3 we shall prove that the second order, as well as the higher order cases, can be recovered by the iterative application of our first order intertwining procedures. Section 3 contains the derivation of a wide set of new n -parametric families of potentials whose spectrum is the same (isospectral), or almost the same (almost isospectral), as the corresponding hydrogen-like spectrum. We shall introduce a set of factorization energies generalizing the choice made for the Infeld-Hull and Mielnik factorizations of the hydrogen-like potentials. As a consequence, the first n energy levels of the n -parametric families can be placed at any previously fixed spectral positions chosen from the above mentioned set of factorization energies. This result opens the possibility of placing *holes* between these levels or between them and the other ones. Some particular cases related to results previously derived by other authors are also mentioned.

2 The Intertwining Method

Let H and \widetilde{H} be the one dimensional Hamiltonians

$$H \equiv -\frac{d^2}{dx^2} + V(x), \quad \widetilde{H} \equiv -\frac{d^2}{dx^2} + \widetilde{V}(x). \quad (1)$$

The n th-order *intertwining method* aims to find a n th-order differential operator A such that

$$\widetilde{H}A = AH. \quad (2)$$

Let H be a Hamiltonian with known solutions to the time-independent Schrödinger equation $H\psi = E\psi$. We are looking for solutions to the corresponding equation for the *intertwined Hamiltonian* \widetilde{H} , $\widetilde{H}\widetilde{\psi} = \widetilde{E}\widetilde{\psi}$. The transformation (2) preserves many spectral properties of the initial Hamiltonian, *e.g.* $\widetilde{H}(A\psi) = E(A\psi)$, $\forall A\psi \neq 0$. We are interested only in those operators A whose action on $L^2(\mathbf{R})$ produces functions in $L^2(\mathbf{R})$; hence we can choose $\widetilde{\psi} \equiv c_0 A\psi$, with c_0 a normalization constant, and we will have $\widetilde{E} = E$, $\forall \widetilde{\psi} = c_0 A\psi \neq 0, \psi \in L^2(\mathbf{R})$. Therefore, the intertwining operator A transforms solutions ψ of the initial Schrödinger equation into solutions $\widetilde{\psi}$ of a new Schrödinger equation, both equations sharing the initial eigenvalues. However, the new Hamiltonian could have a finite number of additional eigenvalues; if this is the case, the corresponding eigenfunctions have to be elements of the kernel of A^\dagger , as we shall see below. We will show now how the intertwining method does work for the simplest case where A is a first order differential operator. This case is specially interesting because it provides a general framework in which various apparently different methods are included.

2.1 The first order intertwining method

Let the operator involved in the intertwining relationship be the following first order differential operator

$$A \equiv a = \frac{d}{dx} + \beta(x). \quad (3)$$

By introducing (1) and (3) in (2) we get

$$-\beta'(x) + \beta^2(x) = V(x) - \epsilon, \quad (4)$$

$$\tilde{V}(x) = V(x) + 2\beta'(x), \quad (5)$$

where ϵ is an arbitrary integration constant, and the prime denotes derivative with respect to x . Notice the implicit dependence of $\beta(x)$ and $\tilde{V}(x)$ on ϵ .

From equations (1) and (5) it becomes apparent that $\tilde{H} - H = 2\beta'(x)$, from (3) we get $[a, a^\dagger] = 2\beta'(x)$, therefore (4) and (5) lead to:

$$H = a^\dagger a + \epsilon, \quad \tilde{H} = aa^\dagger + \epsilon. \quad (6)$$

Equation (6) shows that the first order intertwining relationship (2), with A given by (3), leads in a natural way to the factorization of the intertwined Hamiltonians for each value of $\epsilon \in \mathbf{R}$, provided that $\beta(x)$ satisfies (4) with a given $V(x)$. The traditional approach to the factorization method involves just a particular solution to the Riccati differential equation (4) for a specific value of ϵ [2]. However, it is well known that a particular solution to a Riccati equation allows one to get the corresponding general solution by means of two quadratures [18]. Mielnik's approach takes this into account [13] and gives a term in $\beta(x)$ unnoticed by the traditional factorization method. With the exception of the harmonic oscillator and the hydrogen-like potentials, the general solutions to the corresponding Riccati equation (4) remain unexplored for almost all the Infeld and Hull cases [19, 20]. An interesting discussion on the relationship between Mielnik's method and Darboux transformation was given by Andrianov *et al* in 1985 [11] (see also [20]). On the other hand, equations (4) and (5) are basic in the construction of the SUSY partner Hamiltonians $H^+ \equiv H - \epsilon = a^\dagger a$, and $H^- \equiv \tilde{H} - \epsilon = aa^\dagger$ in supersymmetric quantum mechanics. In this language, it is said that equation (5) relates the two SUSY partner potentials $\tilde{V}(x)$ and $V(x)$, and the function $\beta(x)$ is named the SUSY potential [9]. Notice that the two Hamiltonians H^\pm are positive semi-definite, *i.e.* $\langle H^\pm \rangle \geq 0$; hence we have $E \geq \epsilon$ and $\tilde{E} \geq \epsilon$.

If the eigenfunctions $\{\psi\}$ and eigenvalues $\{E\}$ of H are known, the corresponding eigenfunctions of \tilde{H} are given by $\tilde{\psi} = (E - \epsilon)^{-1/2} a\psi$ with eigenvalues $\tilde{E} = E$. They form an orthonormal set

$$\langle \tilde{\psi}', \tilde{\psi} \rangle = [(E' - \epsilon)(E - \epsilon)]^{-1/2} \langle a\psi', a\psi \rangle = \sqrt{\frac{E' - \epsilon}{E - \epsilon}} \langle \psi', \psi \rangle = 0, \quad \forall \psi' \neq \psi.$$

However, they do not span yet the whole of $L^2(\mathbf{R})$ because the solution to $a^\dagger \tilde{\psi}_\epsilon = 0$ is orthogonal to all the members of the set $\{\tilde{\psi}\}$, $\langle \tilde{\psi}_\epsilon, \tilde{\psi} \rangle = (E - \epsilon)^{-1/2} \langle a^\dagger \tilde{\psi}_\epsilon, \psi \rangle = 0$. That first order linear differential equation is immediately solved:

$$\tilde{\psi}_\epsilon \propto e^{\int \beta(x) dx}. \quad (7)$$

From (6), it is easy to see that $\widetilde{H}\widetilde{\psi}_\epsilon = \epsilon\widetilde{\psi}_\epsilon$. Hence, $\widetilde{\psi}_\epsilon$ is an eigenfunction of \widetilde{H} with eigenvalue ϵ . In the case when $\widetilde{\psi}_\epsilon$ is a square integrable function (*i.e.* $\widetilde{\psi}_\epsilon$ has physical sense as a wavefunction), it has to be added to the set $\{\widetilde{\psi} = (E - \epsilon)^{-1/2}a\psi\}$ in order to complete a new basis on $L^2(\mathbf{R})$. The set of eigenvalues of \widetilde{H} is given by $\{\widetilde{E}\} = \{\epsilon, E\}$, so that the Hamiltonian \widetilde{H} has the same spectrum as H plus a new level at ϵ . Indeed, since the general solution $\beta(x)$ of the Riccati equation (4) has a ‘*free parameter*’ λ , labeling also $\widetilde{V}(x)$ (see equation (5)), the Mielnik’s method allows one to construct, for a specific value of ϵ , a *one*-parametric family of potentials $\widetilde{V}(x)$ which in general will be almost isospectral to $V(x)$.

We are going to discuss now how the Mielnik’s method can be further generalized. First, let us remark that the factorization energy considered by Infeld-Hull and Mielnik is not the only possibility leading to a solvable Riccati equation (4). The key point arises when the non-linear first order differential equation (4) is replaced into a homogeneous linear second order differential equation by means of the transformation¹:

$$\beta(x) = -\frac{d}{dx} \ln u(x), \quad (8)$$

which leads to the second order differential equation:

$$\frac{d^2}{dx^2} u(x) - [V(x) - \epsilon]u(x) = 0. \quad (9)$$

If the factorization energy ϵ belongs to $\{E\}$, then the solutions $u(x)$ to (9) are the same as the previously known eigenfunctions $\psi(x)$ of the initial Hamiltonian H . Following [3], we shall consider just the cases where the factorization energy ϵ does not belong to the spectrum of H . Therefore, the solutions to (9) do not have direct physical meaning, but we have just seen that they naturally lead to the factorization of the Hamiltonians H and \widetilde{H} in the spirit of Mielnik’s approach, providing as well the explicit form for the new potentials $\widetilde{V}(x)$.

Let us remark that n different factorization energies $\epsilon_1, \epsilon_2, \dots, \epsilon_n$, lead to n different Riccati equations (4) and to n *one*-parametric families of potentials $\widetilde{V}_i(x)$, $i = 1, 2, \dots, n$, almost isospectral to $V(x)$. It must be clear that there will be n different factorizations of the initial Hamiltonian H (see equation (6)). With the aim of iterating later the first order intertwining technique let us introduce a slightly different notation. Let us write $\epsilon = \epsilon^{(k)}$, where k labels the specific real number $\epsilon \notin \{E\}$, which we will use as factorization energy. Moreover, we shall write explicitly the dependence of $\beta(x)$ and $\widetilde{V}(x)$ on $\epsilon^{(k)}$ (see equations (4) and (5)) by making $\beta(x) = \beta^{(k)}(x)$ and $\widetilde{V}(x) = V^{(k)}(x)$. For the sake of simplicity the other functions and operators labeled by a *tilde* will be represented as $\widetilde{H} = H^{(k)}$, $\widetilde{E} = E^{(k)}$, etc. Finally, the number of indices will denote the order of the intertwining transformation (2), as well as the number of ‘*free parameters*’ λ which will label the intertwined potential. For example, $H^{(k)}$ represents a first order intertwined Hamiltonian whose corresponding *one*-parametric family of potentials $V^{(k)}(x)$ is almost isospectral to $V(x)$, while $H^{(km)}$ represents a second order intertwined Hamiltonian with

¹The inverse transformation of (8) is $u(x) = \exp(-\int \beta(x)dx)$.

the corresponding *two*-parametric family of potentials $V^{(km)}(x)$ almost isospectral to $V(x)$. The benefits of such notation will be clear in the next sections.

2.2 Second order intertwining method

Let us consider the second order differential operator

$$B^{(km)} = \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \gamma(x), \quad (10)$$

where $\eta(x)$ and $\gamma(x)$ depend implicitly on $\epsilon^{(k)}$ and $\epsilon^{(m)}$. Introducing $H^{(km)} = -d^2/dx^2 + V^{(km)}(x)$ and (10) in the second order intertwining relationship $H^{(km)}B^{(km)} = B^{(km)}H$, with H given in (1), and by ordering at different powers of the operator d/dx , we have for the corresponding coefficients

$$V^{(km)}(x) = V(x) + 2\eta'(x), \quad (11)$$

$$2\gamma(x) = \eta^2(x) - \eta'(x) - d - 2V(x), \quad (12)$$

$$V''(x) + \eta(x)V'(x) = 2\gamma(x)\eta'(x) - \gamma''(x), \quad (13)$$

where d is an integration constant. Following [12], we can express the functions $V(x)$, $V^{(km)}(x)$ and $\gamma(x)$ in terms of $\eta(x)$. The substitution of (12) into (13) produces a third order differential equation which, after multiplying by $\eta(x)$, becomes reduced (by a first integration) to the second order differential equation

$$\eta\eta'' - \frac{(\eta')^2}{2} + \left(2\gamma - \eta' - \frac{\eta^2}{2}\right)\eta^2 + 2c = 0, \quad (14)$$

where we have used once again the equation (12) and c is a new integration constant.

In order to solve the non-linear second order differential equation (14), let us try the ansatz [21]

$$\eta'(x) = \eta^2(x) + z(x)\eta(x) + b, \quad (15)$$

where the constant b and the function $z(x)$ are to be determined. Introducing (15) in (14) we get $b_{\pm} = \pm 2\sqrt{c}$, $c > 0$, besides two Riccati differential equations

$$z'_{\pm}(x) + \frac{z_{\pm}^2(x)}{2} - 2V(x) - d - b_{\pm} = 0. \quad (16)$$

The identification $z_+(x) = -2\beta_k(x)$, $z_-(x) = -2\beta_m(x)$, and

$$d = -\epsilon^{(m)} - \epsilon^{(k)}, \quad c = \left(\frac{\epsilon^{(m)} - \epsilon^{(k)}}{2}\right)^2, \quad \epsilon^{(m)} \neq \epsilon^{(k)}, \quad (17)$$

allows one to write the solutions of (14) in the form

$$\eta(x) = -\left(\frac{\epsilon^{(m)} - \epsilon^{(k)}}{\beta^{(m)}(x) - \beta^{(k)}(x)}\right), \quad \epsilon^{(m)} \neq \epsilon^{(k)}, \quad (18)$$

where $\beta^{(m)}(x)$ and $\beta^{(k)}(x)$ are both solutions to (4) with factorization energies $\epsilon^{(m)}$ and $\epsilon^{(k)}$ respectively. Equations (18) and (12) determine the second order intertwining operator $B^{(km)}$. Notice that, by means of this process, we can derive solutions to the non-linear second order differential equation (14) by just solving the easier non-linear first order differential equation (4) for *two different* values of the factorization energy ϵ .

The action of $B^{(km)}$ on the eigenfunctions $\{\psi\}$ of H gives the basic set of eigenfunctions $\{\Psi^{(km)} \propto B^{(km)}\psi\}$ of the intertwined Hamiltonian $H^{(km)}$. On the other hand, the solutions to the second order differential equation $B^{(km)\dagger}\Psi_B^{(km)} = 0$ are orthogonal to all the $\Psi^{(km)} \propto B^{(km)}\psi$. Now, because the kernel of $B^{(km)\dagger}$ is a 2-dimensional subspace, the maximum number of linearly independent elements is two. Therefore, we are looking for two square integrable linearly independent solutions to $B^{(km)\dagger}\Psi_B^{(km)} = 0$ such that they are simultaneously eigenfunctions of $H^{(km)}$, with eigenvalues to be determined.

Let us now write $\Psi_B^{(km)} = \mathcal{C}_0 \exp(\int f^{(km)}(x)dx)$ as a generic kernel element of $B^{(km)\dagger}$; then the equation $B^{(km)\dagger}\Psi_B^{(km)} = 0$ can be rewritten as

$$\frac{d}{dx}f^{(km)}(x) - \eta(x)f^{(km)}(x) + (f^{(km)}(x))^2 - \eta'(x) + \gamma(x) = 0. \quad (19)$$

Using equations (4), (12) and (15), the general solutions to (19) can be obtained:

$$\Psi_B^{(km)}(x) = \mathcal{C}_k \frac{u^{(k)}(x)}{W(k, m)} + \mathcal{C}_m \frac{u^{(m)}(x)}{W(k, m)}, \quad (20)$$

where $u^{(k)}(x)$ and $u^{(m)}(x)$ are the two *unphysical* solutions to (9) with eigenvalues $\epsilon^{(k)}$ and $\epsilon^{(m)}$, respectively. The function $W(k, m) = u^{(k)}(u^{(m)})' - (u^{(k)})'u^{(m)}$ is the wronskian of $u^{(k)}(x)$ and $u^{(m)}(x)$.

By using (8) and (9) we now rewrite (18) as

$$\eta(x) = (\epsilon^{(m)} - \epsilon^{(k)}) \frac{u^{(k)}(x)u^{(m)}(x)}{W(k, m)} = -\frac{d}{dx} \ln W(k, m). \quad (21)$$

Notice that the right hand side of (21) corresponds to the Crum's determinant [17]. Hence, the coefficient of the second term of $B^{(km)}$ in (10) can be constructed either by using (21) or simply by (18). Using now (16), (8) and (9), with $\eta(x)$ as given above, it is easy to show that the function $u^{(k)}/W(k, m)$ is eigenfunction of $H^{(km)}$ with eigenvalue $\epsilon^{(m)}$. A similar procedure shows that $u^{(m)}/W(k, m)$ is also an eigenfunction of $H^{(km)}$ with eigenvalue $\epsilon^{(k)}$. Taking this into account we write

$$\Psi_{\epsilon_m}^{(km)}(x) \propto \frac{u^{(k)}(x)}{W(k, m)}; \quad \Psi_{\epsilon_k}^{(km)}(x) \propto \frac{u^{(m)}(x)}{W(k, m)}, \quad (22)$$

where the subindex ϵ_m (ϵ_k) indicates the corresponding eigenvalue $\epsilon^{(m)}$ ($\epsilon^{(k)}$). Therefore, the eigenfunctions of $H^{(km)}$ are $\{\Psi_{\epsilon_m}^{(km)}, \Psi_{\epsilon_k}^{(km)}\} \cup \{\Psi^{(km)} \propto B^{(km)}\psi \mid B^{(km)}\psi \neq 0\}$, and the corresponding eigenvalues are $\{E^{(km)}\} = \{\epsilon^{(m)}, \epsilon^{(k)}, E\}$. Thus, $H^{(km)}$ has the same spectrum as H plus two new levels at $\epsilon^{(m)}$ and $\epsilon^{(k)}$, provided that $\Psi_{\epsilon_m}^{(km)}$ and $\Psi_{\epsilon_k}^{(km)}$ are square

integrable functions. As (18) involves two general solutions to the Riccati equation (4), thus $V^{(km)}(x)$ represents a set of *two*-parametric families of potentials almost isospectral to $V(x)$.

Notice that we have not placed any ordering to the levels $\epsilon^{(m)}$ and $\epsilon^{(k)}$ so that it is possible either that $\epsilon^{(m)} > \epsilon^{(k)}$ or that $\epsilon^{(m)} < \epsilon^{(k)}$. This is implicit in the fact that $\eta(x)$ in (18) is invariant under the change $k \leftrightarrow m$. This will be important in the next section, where we are going to show that the results found by means of the second order intertwining technique can be found also by the iteration of two successive first order intertwining transformations.

2.3 Iterative factorization

Let us come back to the results of Section 2.1 and suppose that by means of the first order intertwining technique we have obtained a Hamiltonian $H^{(k)}$ from H using a certain factorization energy $\epsilon^{(k)}$. We look now for a new Hamiltonian $h^{(km)} = -d^2/dx^2 + v^{(km)}(x)$ attainable from $H^{(k)}$ by means of the first order intertwining method, *i.e.*, $h^{(km)}$ and $H^{(k)}$ satisfy the following relationship

$$h^{(km)} a^{(km)} = a^{(km)} H^{(k)},$$

where $a^{(km)}$ is the first order differential operator $a^{(km)} \equiv d/dx + \beta^{(km)}(x)$. As usual, we arrive at the standard equations linking $v^{(km)}(x)$, $V^{(k)}(x)$ and $\beta^{(km)}(x)$:

$$-(\beta^{(km)})' + (\beta^{(km)})^2 = V^{(k)}(x) - \varepsilon_m = 2(\beta^{(k)})' + V(x) - \varepsilon_m, \quad (23)$$

$$v^{(km)}(x) = V^{(k)}(x) + 2(\beta^{(km)})'(x) = V(x) + 2[\beta^{(km)}(x) + \beta^{(k)}(x)]', \quad (24)$$

where ε_m is a new integration constant (factorization energy) and we have used equation (5). The intertwined Hamiltonians once again become factorized

$$H^{(k)} = a^{(km)\dagger} a^{(km)} + \varepsilon_m, \quad h^{(km)} = a^{(km)} a^{(km)\dagger} + \varepsilon_m.$$

The eigenvalues of $h^{(km)}$ are given by $\{\mathcal{E}^{(km)}\} = \{\varepsilon_m, E^{(k)}\}$, and the eigenfunctions by $\psi_{\varepsilon_m}^{(km)} \cup \{\psi^{(km)} = (E^{(k)} - \varepsilon_m)^{-1/2} a^{(km)} \psi^{(k)} \mid a^{(km)} \psi^{(k)} \neq 0\}$, where the missing state $\psi_{\varepsilon_m}^{(km)} \propto \exp(\int \beta^{(km)}(x) dx)$ corresponds to the eigenvalue ε_m .

The key point here becomes to find the general solution to the new Riccati's differential equation (23) for some $\varepsilon_m \in \mathbf{R}$. The obvious solution $(\beta^{(km)})'(x) = -(\beta^{(k)})'(x)$, for which $\varepsilon_m = \epsilon^{(k)}$, corresponds to a Hamiltonian $h^{(km)}$ equal to H because $v^{(km)}(x) = V(x)$. Avoiding this trivial solution we shall consider $\varepsilon_m \neq \epsilon^{(k)}$. Non-trivial solutions to (23) can be found by inspecting the following intertwining relationship:

$$h^{(km)} a^{(km)} a^{(k)} = a^{(km)} H^{(k)} a^{(k)} = a^{(km)} a^{(k)} H. \quad (25)$$

Equation (25) shows that $h^{(km)}$ and H are related by the product of two first order intertwining operators, *i.e.*, by the second order differential operator $C^{(km)} \equiv a^{(km)} a^{(k)}$. An interesting question immediately arises: is the Hamiltonian $h^{(km)}$ the same as the

$H^{(km)}$ of Section 2.2? If $C^{(km)} = B^{(km)}$ the answer is affirmative, and in this case the SUSY potential $\beta^{(k)}(x)$, the *iterated* new SUSY potential $\beta^{(km)}(x)$, and the functions $\gamma(x)$ and $\eta(x)$ are related by the equations

$$\eta(x) = \beta^{(km)}(x) + \beta^{(k)}(x), \quad \gamma(x) = \frac{d}{dx}\beta^{(km)}(x) + \beta^{(km)}(x)\beta^{(k)}(x). \quad (26)$$

Comparing (18) and (26) we get the solutions of (23) (with $\varepsilon_m = \epsilon^{(m)}$):

$$\beta^{(km)}(x) = -\beta^{(k)}(x) - \left(\frac{\epsilon^{(m)} - \epsilon^{(k)}}{\beta^{(m)}(x) - \beta^{(k)}(x)} \right), \quad (27)$$

where $\beta^{(k)}(x)$ ($\beta^{(m)}(x)$) is a solution of (4) associated to the factorization energy $\epsilon^{(k)}$ ($\epsilon^{(m)}$). Equation (27) is nothing but the theorem of Fernández *et al* [15]. The eigenfunctions of $h^{(km)} = H^{(km)}$ are given by the basic set

$$\Psi_{\epsilon_k}^{(km)} = (\epsilon^{(k)} - \epsilon^{(m)})^{-1/2} a^{(km)} \psi_{\epsilon_k}^{(k)}, \quad \Psi^{(km)} = (E - \epsilon^{(m)})^{-1/2} (E - \epsilon^{(k)})^{-1/2} a^{(km)} a^{(k)} \psi,$$

plus the new missing state

$$\Psi_{\epsilon_m}^{(km)} \propto e^{\int \beta^{(km)}(x) dx},$$

which, by definition, is such that $a^{(km)\dagger} \Psi_{\epsilon_m}^{(km)} = 0$. Hence, by construction $B^{(km)\dagger} \Psi_{\epsilon_m}^{(km)} = a^{(k)\dagger} a^{(km)\dagger} \Psi_{\epsilon_m}^{(km)} = 0$. On the other hand, remembering that $a^{(k)\dagger} \psi_{\epsilon_k}^{(k)} = 0$, we get

$$B^{(km)\dagger} \Psi_{\epsilon_k}^{(km)} = (\epsilon^{(k)} - \epsilon^{(m)})^{-1/2} a^{(k)\dagger} [H^{(km)} - \epsilon^{(m)}] \psi_{\epsilon_k}^{(k)} = 0.$$

Now, by using (26), (8) and after (21), we can write

$$\Psi_{\epsilon_k}^{(km)}(x) = (\epsilon^{(k)} - \epsilon^{(m)})^{1/2} \frac{u^{(m)}(x)}{W(k, m)},$$

which corresponds to the definition of $\Psi_{\epsilon_k}^{(km)}(x)$ given in (22). A similar procedure but now using (27), (8) and (21), allows one to recover the corresponding expression for $\Psi_{\epsilon_m}^{(km)}(x)$. Finally, the set of eigenvalues of $h^{(km)} = H^{(km)}$ is given by $\{\mathcal{E}^{(km)} = E^{(km)}\} = \{\epsilon^{(m)}, \epsilon^{(k)}, E\}$, just as we have shown in Section 2.2.

It is worth to notice that the operator $B^{(km)} = a^{(km)} a^{(k)}$ does not factorize the intertwined Hamiltonians H and $H^{(km)}$, but some of their quadratic forms:

$$\begin{aligned} B^{(km)\dagger} B^{(km)} &= a^{(k)\dagger} a^{(km)\dagger} a^{(km)} a^{(k)} = a^{(k)\dagger} [H^{(k)} - \epsilon^{(m)}] a^{(k)} = a^{(k)\dagger} a^{(k)} [H - \epsilon^{(m)}] \\ &= (H - \epsilon^{(k)})(H - \epsilon^{(m)}) = (H - \epsilon^{(m)})(H - \epsilon^{(k)}), \end{aligned} \quad (28)$$

and

$$B^{(km)} B^{(km)\dagger} = (H^{(km)} - \epsilon^{(m)})(H^{(km)} - \epsilon^{(k)}). \quad (29)$$

Up to now, from the general solution to (4) for each factorization energy $\epsilon^{(k)}$, we have derived a *one*-parametric family of potentials $V^{(k)}(x)$ isospectral to $V(x)$ but by a new level at $\epsilon^{(k)}$. From two of these solutions with $\epsilon^{(k)} \neq \epsilon^{(m)}$ we have derived as well

two-parametric families isospectral to $V(x)$ but by two new levels at $\epsilon^{(m)}$ and $\epsilon^{(k)}$. It is possible to iterate further the first order method by considering now the general solutions to (4), associated to three different factorization energies $\epsilon^{(k)}$, $\epsilon^{(m)}$, $\epsilon^{(l)}$. The key Riccati equation to be solved becomes now:

$$-(\beta^{(kml)})'(x) + (\beta^{(kml)})^2(x) = V^{(km)}(x) - \epsilon^{(l)}.$$

The algorithm (27) allows one to write $\beta^{(kml)}(x)$ in terms of two different solutions of (23),

$$\beta^{(kml)}(x) = -\beta^{(km)} - \left(\frac{\epsilon^{(l)} - \epsilon^{(m)}}{\beta^{(kl)}(x) - \beta^{(km)}(x)} \right),$$

which leads to the following 3-parametric family of potentials $V^{(kml)}(x)$:

$$V^{(kml)}(x) = V^{(km)}(x) + 2(\beta^{(kml)})'(x) = V(x) + 2[\beta^{(kml)}(x) + \beta^{(km)}(x) + \beta^{(k)}(x)], \quad (30)$$

having the same spectrum as $V(x)$ but by three new energy levels at $\epsilon^{(l)}$, $\epsilon^{(m)}$ and $\epsilon^{(k)}$ [15]. The corresponding third order intertwining operator is given by $C^{(kml)} = a^{(kml)}a^{(km)}a^{(k)}$. This leads to the factorization of some six order differential operators, related to the two intertwined Hamiltonians by means of:

$$\begin{aligned} C^{(kml)\dagger}C^{(kml)} &= (H - \epsilon^{(l)})(H - \epsilon^{(m)})(H - \epsilon^{(k)}) \\ C^{(kml)}C^{(kml)\dagger} &= (H^{(kml)} - \epsilon^{(l)})(H^{(kml)} - \epsilon^{(m)})(H^{(kml)} - \epsilon^{(k)}). \end{aligned}$$

This process can be continued at will, and the corresponding formulae for the key functions can be algebraically obtained from the solutions $\beta^{(k)}(x)$ to (4). It is interesting to see now how the process does work explicitly for a physically meaningful system.

3 The hydrogen-like potentials

Let us consider a single electron in the field produced by a nucleus with charge Ze , where Z is the number of protons in the nucleus². The time-independent Schrödinger equation is given by:

$$-\nabla^2\Psi(\vec{r}) - \frac{2}{r}\Psi(\vec{r}) = E\Psi(\vec{r}), \quad (31)$$

where we are using the units of $r_B = \hbar^2/Ze^2m$, and $\mathcal{E} = Z/2r_B$ for the coordinate and energy respectively. Due to the spherical symmetry, the standard choice $\Psi(\vec{r}) = R(r)Y(\theta, \phi)$ separates (31) into a radial and an angular equation. The angular solutions are given by the spherical harmonics $Y_l^m(\theta, \phi)$ and the energy spectrum can be finally obtained by solving the radial equation:

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2}{r} \right] \psi(r) = E\psi(r), \quad (32)$$

² $Z = 1$ describes the Hydrogen atom, while ions or atoms with only one electron in the outermost shell can be considered as hydrogen-like systems with $Z > 1$.

where, $l = 0, 1, 2, \dots$, is the azimuthal quantum number and, for the sake of simplicity, we will work with $\psi(r) \equiv rR(r)$, $0 \leq r < +\infty$. The set of solutions of (32) span the whole of $L^2(\mathbf{R}^+)$, with an inner product defined by $\langle \psi, \psi' \rangle = 4\pi \int_0^{+\infty} \psi(r)\psi'(r)dr < \infty$. The differential operator of the left hand side of (32) will be referred as the radial Hamiltonian, and it will be denoted by H_l , where

$$H_l = -\frac{d^2}{dr^2} + V_l(x); \quad V_l(x) \equiv \frac{l(l+1)}{r^2} - \frac{2}{r}. \quad (33)$$

The eigenvalues, for a fixed l , are given by the well known formula

$$E_n \equiv E_{lK} = -\frac{1}{(l+K)^2}, \quad K = 1, 2, 3, \dots \quad (34)$$

3.1 The new one-parametric families

Let us consider the first order intertwining relationship $H_{l-1}^{(k)}a_l^{(k)} = a_l^{(k)}H_l$ where, as a result of the further calculations, we are labeling from the beginning the unknown Hamiltonian $H_{l-1}^{(k)}$ with the subindex $l-1$ because it will arise a centrifugal term for the new potential with exactly that index. In order to solve the corresponding Riccati equation (4) we shall use its equivalent second order differential equation (9) rewritten as

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2}{r} \right] u_l^{(k)}(r) = \epsilon_l^{(k)} u_l^{(k)}(r). \quad (35)$$

Comparing (35) with (32) and (34), it is natural to propose the factorization energies as:

$$\epsilon_l^{(k)} \equiv -\frac{1}{(l+k)^2}, \quad k \neq K, \quad l > 0, \quad (36)$$

where, in general, k could be either a discrete as well as a continuous real number. On the other hand, note that the case $k = K$ reproduces the well known solutions to the *physical* eigenproblem (32-34); hence, following Mielnik we have taken $k \neq K$. It is clear that, for a fixed value of l , we have $\epsilon_l^{(k)} \neq E_{lK}$, $\forall k \neq K$.

In order to solve (35) we make the transformation

$$u_l^{(k)}(r) = r^{-l} e^{r/(l+k)} \Phi_l^{(k)}(r), \quad (37)$$

leading to a confluent hypergeometric equation for $\Phi_l^{(k)}(r)$, whose general solution for the discrete values $k = 0, -1, -2, \dots, -(l-1)$ is given by the linear combination of confluent hypergeometric functions [22, 23]:

$$\Phi_l^{(k)}(r) = {}_1F_1[k, -2l, -2r/(l+k)] - \nu_{lk} [2r/(l+k)]^{1+2l} {}_1F_1[1+k+2l, 2+2l, -2r/(l+k)], \quad (38)$$

with

$$\nu_{lk} \equiv \frac{\Gamma(1+|k|)}{\Gamma(2l+2)} \frac{\lambda_l^{(k)}}{(-2l)_{|k|}}, \quad l > 0, \quad (39)$$

where $\lambda_l^{(k)}$ is a constant to be determined and

$$(-2l)_{|k|} \equiv \frac{\Gamma(-2l + |k|)}{\Gamma(-2l)} = (-2l + |k| - 1)(-2l + |k| - 2) \cdots (-2l).$$

Note that $\Phi_l^{(k)}(r=0) = 1$, while its asymptotic behaviour is given by

$$\Phi_l^{(k)}(r) \sim \left(\frac{2r}{l - |k|} \right)^{|k|} \frac{1 - \lambda_l^{(k)}}{(-2l)_{|k|}}.$$

Hence, the solutions (37) are divergent at the origin as well as in the limit $r \rightarrow \infty$, and it is clear that $u_l^{(k)}(r) \notin L^2(\mathbf{R}^+)$. This is the reason why we consider them as *unphysical* solutions to the Schrödinger equation (32) with the *atypical* eigenvalues (36). Let us remark that the discreteness of k in (37-39) leads to the most transparent case available with the mathematical tools at hand [22]. The general solution to the corresponding Riccati equation (4) arises after introducing (37) in (8), which gives:

$$\beta_l^{(k)}(r) = \frac{l}{r} - \frac{1}{(l+k)} - \frac{d}{dr} \ln \Phi_l^{(k)}(r), \quad l > 0. \quad (40)$$

Using (5) and (40), we get the first order intertwined potentials

$$V_{l-1}^{(k)}(r) = V_{l-1}(r) - 2 \frac{d^2}{dr^2} \ln \Phi_l^{(k)}(r), \quad l > 0. \quad (41)$$

The second term of (41) is free of singularities if

$$\lambda_l^{(k)} \in \begin{cases} (-\infty, 1), & \text{for } |k| \text{ even;} \\ (1, \infty), & \text{for } |k| \text{ odd.} \end{cases} \quad (42)$$

Hence, the new potentials $V_{l-1}^{(k)}(r)$ have the same singularity at $r = 0$ as $V_{l-1}(r)$, provided that $\lambda_l^{(k)}$ takes only values in the domain (42). Since the second term of $V_{l-1}^{(k)}(r)$ tends to zero when $r \rightarrow 0$ and $r \rightarrow \infty$, we conclude that $V_{l-1}^{(k)}(r)$ behaves as $V_{l-1}(r)$ at the ends of $[0, \infty)$.

Now, the eigenfunctions of $H_{l-1}^{(k)}$ are obtained from the action of $a_l^{(k)}$ on the eigenfunctions $\{\psi_{nl}(r)\}$ of H_l , $\psi_{n,l-1}^{(k)} = (E_n - \epsilon_l^{(k)})^{-1/2} a_l^{(k)} \psi_{nl}(r)$, plus the *isolated* ground state:

$$\psi_{l-1,\epsilon_k}^{(k)} = C_{lk} \frac{r^l e^{-r/(l+k)}}{\Phi_l^{(k)}(r)}, \quad (43)$$

where $\Phi_l^{(k)}(r)$ is given by (38), and

$$C_{lk} = \left[\left(\frac{2}{l - |k|} \right)^{2l+1} \left(\frac{1 - \lambda_l^{(k)}}{(-2l)_{|k|}} \right) \frac{|k|!}{(2l)!} \right]^{1/2}, \quad k = 0, -1, -2, \dots, -(l-1). \quad (44)$$

It is important to notice that, despites the resemblance between the missing states (43) and the *unphysical* functions (37), $\psi_{l-1,\epsilon_k}^{(k)} \sim 1/u_l^{(k)}$, just the first ones become square integrable functions. The eigenvalues of $H_{l-1}^{(k)}$ (for fixed l and k) are:

$$E_{l-1,K}^{(k)} = -\frac{1}{(l+k)^2}, -\frac{1}{(l+K)^2}, \quad K = 1, 2, \dots \quad (45)$$

By comparing with (34) it is apparent that our families of potentials (41) have, in general, the same spectrum as the corresponding spectrum of the hydrogen-like potentials $V_{l-1}(r)$, plus a new level at $\epsilon_l^{(k)}$.

3.2 Particular one-parametric families

Notice that, for fixed l , k can take l different values $k = 0, -1, -2, \dots, -(l-1)$. Hence, there are l different factorization energies $\epsilon_l^{(k)}$ generating l non-equivalent families of solvable potentials $V_{l-1}^{(k)}(r)$. In particular, let us notice that $\epsilon_l^{(0)} = -1/l^2 \equiv L(l)$, with $\lambda_l^{(k)} = 0$, lead to the standard factorization of the hydrogen-like potentials as presented by Infeld-Hull (see equation 8.1.3, and page 68 of [2]). Thus, our technique can be considered as a generalization of the Infeld-Hull factorization where, instead of looking for particular solutions to (4) with $\epsilon_l^{(0)} = L(l)$, we look for the general solutions with $\epsilon_l^{(k)} = L(l+k)$. In a similar way, our results can be seen as a generalization of those derived by Fernández, who found a general solution to (4) but just in the case with $k = 0$ [4]. Hence, for a fixed l and taking $k = 0$, the equation (45) leads to $E_{l-1,K}^{(0)} = E_{l-1,K}$, *i.e.*, $V_{l-1}^{(0)}(r)$ is a family of potentials strictly isospectral to $V_{l-1}(r)$. On the other hand, for $k \neq 0$, we have not only $\epsilon_{l-1}^{(k)} \neq E_{l-1,K}$, but $\epsilon_{l-1}^{(k)} < E_{l-1,K=1} = -1/(l+1)^2$. In this case the spectrum of $H_{l-1}^{(k \neq 0)}$ is almost the same as the one of H_{l-1} , the difference resting in the ground state energy level. It is clear now that the functions (41) represent a set of $l-1$ families of potentials almost isospectral to $V_{l-1}(r)$, plus a family strictly isospectral to $V_{l-1}(r)$.

3.2.1 The case $k = 0$

The member of the family (41), for $k = 0$, is given by

$$V_{l-1}^{(0)}(r) = V_{l-1}(r) - 2\frac{d^2}{dr^2} \ln \Phi_l^{(0)}(r), \quad l > 0, \quad (46)$$

where

$$\Phi_l^{(0)}(r) = 1 - \left(\frac{2}{l}\right)^{2l+1} \frac{\lambda_l^{(0)}}{(2l)!} \int_0^r x^{2l} e^{-2x/l} dx. \quad (47)$$

From (42) we see that $V_{l-1}^{(0)}(r)$ does not have more singularities than $V_{l-1}(r)$ if $\lambda_l^{(0)} \in (-\infty, 1)$. In particular, for $\lambda_l^{(0)} = 0$, we have $\Phi_l^{(0)}(r) = 1$ and $V_{l-1}^{(0)}(r) = V_{l-1}(r)$, which means that $V_{l-1}(r)$ itself is a member of the family (46). On the other hand, we get also the standard SUSY potential $W(r) \equiv \beta_l^{(0)}(r) = l/r - 1/l$, besides the corresponding first order operator $a_l^{(0)} = d/dr + l/r - 1/l$, both of them used to solve the hydrogen-like Hamiltonians

[1, 2, 9]. Let us remark once again that $V_{l-1}^{(0)}(r)$ with $\lambda_l^{(0)} = (2l)!(l/2)^{2l+1}\gamma_l^{-1}$ leads to the *one*-parametric family of isospectral hydrogen-like potentials derived by Fernández [4]. If $l = 1$ and $\lambda_1^{(0)} \rightarrow 1$, the potential (46) tends also to the one found by Abraham and Moses [24].

3.2.2 The case $k = -1$

Let us take now $k = -1$. Thus, the potential (41) becomes

$$V_{l-1}^{(-1)}(r) = V_{l-1}(r) - 2\frac{d^2}{dr^2} \ln \Phi_l^{(-1)}(r), \quad l > 1, \quad (48)$$

where

$$\Phi_l^{(-1)}(r) = \left[1 - \frac{r}{l(l-1)}\right] \left\{1 + \frac{\lambda_l^{(-1)}}{(2l-1)!} \left(\frac{2}{l-1}\right)^{2l-1} \int_0^r \frac{x^{2l} e^{-2x/(l-1)}}{[l(l-1)-x]^2} dx\right\}. \quad (49)$$

In this case $\lambda_l^{(-1)} \in (1, \infty)$, and although $V_{l-1}(r)$ governs the behaviour at the origin and at infinity of $V_{l-1}^{(-1)}(r)$, it is not a member of the family (48). The corresponding eigenvalues are given by

$$E_{l-1,K}^{(-1)} = -\frac{1}{(l-1)^2}, -\frac{1}{(l+1)^2}, -\frac{1}{(l+2)^2}, \dots, \quad l > 1. \quad (50)$$

Notice the ‘jump’ between the first two energy levels of $H_{l-1}^{(-1)}$; this produces a *hole* in the spectrum (50) because the level at $-1/l^2$ is absent. Comparing (50) with (34) it turns out that $V_{l-1}^{(-1)}(r)$ represents a family of potentials almost isospectral to $V_{l-1}(r)$. As a particular example let us consider $l = 2$: in this case $V_1^{(-1)}(r)$ has the same levels, $-1/n^2$, as $V_1(r)$ for $n \geq 3$. However, the ground state energy level of $V_1^{(-1)}(r)$ is at -1 , which is forbidden for $V_1(r)$ whose ground state is at $-1/4$. The potentials (48-49) have been recently reported by the author (see equations (10) and (22) of [14]) and they are plotted in Figure 1 for $l = 2$ and different values of $\lambda_2^{(-1)}$.

3.2.3 The case $|k| > 1$

For $|k| > 1$, there is a gap between the two first energy levels of $H_{l-1}^{(k)}$, which grows up as $|k|$ increases. For instance, when $|k| = 2$, and $l = 3$, the levels labeled by $n = 2$ and $n = 3$ are not present in the spectrum of $H_2^{(-2)}$. A similar result, but with a different method and Hamiltonian, has been derived by Samsonov [11].

Recently, the SUSY potential $\omega(x) = a/\gamma - \gamma/x$, with $a > 0$ and $\gamma > 0$, has been used to construct a ‘conditionally exactly solvable (CES) potential’ $V_+(x)$ for the Hydrogen atom problem [25, 26]. In such a paper a is a coupling constant while γ is a parameter taking some strange values as $\gamma = 2.8$ (see Fig. 7 in [26]). Although there is a close relationship between that potential $V_+(x)$ and our first order intertwined potential $V_{l-1}^{(k)}(r)$, it is important some caution:

i) If $\gamma > 0$ is not an integer, the SUSY potential $\omega(x)$ of [26] is not related to the physically relevant SUSY potential $W(r)$ for the Hydrogen atom, and in this case $V_+(x)$ is the ‘CES potential’ for some *mathematical problem*. If the so called CES potentials derived in [26] are to be related with a “physical potential”, the condition $\gamma > 0$ is not sufficient.

ii) If γ is an integer greater than zero, the SUSY potential $\omega(x)$ can be related to the Hydrogen atom problem. In this case the function (5.14) of [26] has a term ${}_1F_1(-\gamma - a/\rho, -2\gamma, 2\rho x)$, which is meaningless at $2\rho x = 0$ unless $\gamma + a/\rho$ is an integer greater than or equal to zero [23]. Indeed, since γ is a positive integer, the fraction a/ρ is an integer such that $\gamma \geq -a/\rho$. Hence, the condition $a > 0$ is neither sufficient to ensure the generality of the potential $V_+(x)$.

Taking this into account, a general ‘CES potential’ $V_+(x)$ really related to the Hydrogen atom problem would take for instance $-a/\rho = \gamma + k$, where k is an integer such that $k \leq 0$. In particular, for $\gamma = l$, and $x = r/a$, the CES potential $V_+(x)$ derived in [26] would be equal to our first order intertwined potential $V_{l-1}^{(k)}(r)$, provided that $\nu_{lk} = \beta/\alpha$, where α and β are constants.

We are going to derive next some *two*-parametric families by means of the technique discussed in sections 2.2 and 2.3 for the particular case we are dealing with.

3.3 The new two-parametric families

In Section 2.2 we have seen that the second order intertwining relationship $H_{l-2}^{(km)} B_{l-2}^{(km)} = B_{l-2}^{(km)} H_l$ provides new solvable potentials if we are able to find some solutions $\eta(x)$ to equation (11). Then we also have shown that the general form of that solutions is given by:

$$\eta(r) = (\epsilon_l^{(k)} - \epsilon_l^{(m)}) \left\{ \frac{d}{dr} \ln \left(\frac{u_l^{(k)}(r)}{u_l^{(m)}(r)} \right) \right\}^{-1}, \quad \epsilon_l^{(k)} \neq \epsilon_l^{(m)}. \quad (51)$$

For our particular system we take the functions $u_l^{(k)}(r)$ as given by (37). The behaviour of our $\eta(r)$ at the ends of $[0, \infty)$ is given by

$$\eta(r)_{r \rightarrow \infty} \sim -\frac{2l + k + m}{(l + k)(l + m)}, \quad \eta(r)_{r \rightarrow 0} \sim -\frac{(1 - 2l)}{r}.$$

Hence, it is natural to write the new potential $V_{l-2}^{(km)}(r)$ as (see equation (11)):

$$V_{l-2}^{(km)}(r) = V_{l-2}(r) + 2\alpha'(r), \quad (52)$$

where $\alpha(r) \equiv \eta(r) + (1 - 2l)/r$ is an appropriate function making evident the limit $V_{l-2}^{(km)}(r) \rightarrow V_{l-2}(r)$ when $r \rightarrow +\infty$, or $r \rightarrow 0$. The *two*-parametric domain of $\lambda_l^{(k)}$ and $\lambda_l^{(m)}$, for which $\alpha'(r)$ is free of singularities, is determined by the parities of k and m . In order to make transparent the choice of that domain we take the following convention: n

is the index labelling the factorization energy $\epsilon_l^{(n)}$ defined by $\epsilon_l^{(n)} \equiv \max(\epsilon_l^{(k)}, \epsilon_l^{(m)})$, and s is such that $\epsilon_l^{(s)} \equiv \min(\epsilon_l^{(k)}, \epsilon_l^{(m)})$. Hence, we have:

$$\begin{cases} \lambda_l^{(n)} \in (-\infty, 1), & \lambda_l^{(s)} \in (-\infty, 1) ; & |n| \text{ even}, |s| \text{ odd} \\ \lambda_l^{(n)} \in (1, \infty), & \lambda_l^{(s)} \in (1, \infty) ; & |n| \text{ odd}, |s| \text{ even}. \end{cases} \quad (53)$$

$$\begin{cases} \lambda_l^{(n)} \in (-\infty, 1), & \lambda_l^{(s)} \in (1, \infty) ; & |n| \text{ even}, |s| \text{ even} \\ \lambda_l^{(n)} \in (1, \infty), & \lambda_l^{(s)} \in (-\infty, 1) ; & |n| \text{ odd}, |s| \text{ odd} \end{cases} \quad (54)$$

The eigenfunctions of $H_{l-2}^{(km)}$ are given by

$$\Psi_{n,l-2}^{(km)}(r) = \frac{B_{l-2}^{(km)} \psi_{nl}(r)}{\sqrt{(E_n - \epsilon_l^{(m)})(E_n - \epsilon_l^{(k)})}}, \quad (55)$$

and

$$\Psi_{\epsilon_k, l-2}^{(km)}(r) = \mathcal{C}_{lk} \sqrt{\epsilon_l^{(k)} - \epsilon_l^{(m)}} \frac{u_l^{(m)}(r)}{W(k, m)}, \quad \Psi_{l-2, \epsilon_m}^{(km)}(r) = \mathcal{C}_{lm} \sqrt{\epsilon_l^{(m)} - \epsilon_l^{(k)}} \frac{u_l^{(k)}(r)}{W(m, k)}, \quad (56)$$

where \mathcal{C}_{lk} is given by (44). The corresponding eigenvalues are:

$$E_{l-2, K}^{(km)} = \left\{ -\frac{1}{(l+m)^2}, -\frac{1}{(l+k)^2} \right\} \cup \left\{ -\frac{1}{(l+K)^2} \right\}, \quad K > 0. \quad (57)$$

Let us remark that the only restriction for the factorization energies, $\epsilon_l^{(k)} \neq \epsilon_l^{(m)}$, leads to the fact that $\eta(r)$ and $V_{l-2}^{(km)}(r)$ are symmetric under the change $k \rightarrow m$ and *vice versa*, $V_{l-2}^{(km)}(r) = V_{l-2}^{(mk)}(r)$. However, this symmetry is broken for the intermediate *one-parametric* potentials arising when $V_{l-2}^{(km)}(r)$ is obtained after two iterated first order intertwining procedures. In order to see that, let us suppose that $\epsilon_l^{(k)} > \epsilon_l^{(m)}$. We have two different ways to get the potentials (52): we can add first the level $\epsilon_l^{(k)}$, and then the level $\epsilon_l^{(m)}$; the other option is to perform the two operations in the opposite order.

(I) In the first case, after the first step we shall get the Hamiltonian $H_{l-1}^{(k)}$, with a spectrum equal to the spectrum of H_l plus a new level at $\epsilon_l^{(k)} \leq E_{l, K=1}$, where $E_{l, K=1} = -1/(l+1)^2$ is the ground state energy level of H_l . After the second step, the final Hamiltonian $H_{l-2}^{(km)}$ has the same spectrum as $H_{l-1}^{(k)}$ plus a new level at $\epsilon_l^{(m)} < \epsilon_l^{(k)}$. In both of these procedures the factorization energy (the ground state energy of the new Hamiltonian) is less than the ground state energy of the initial Hamiltonian, just as it happens in the SUSY approach.

(II) Taking now the two operations in the opposite order, the first intertwined Hamiltonian $H_{l-1}^{(m)}$ has a spectrum equal to the one of H_l plus a new level at $\epsilon_l^{(m)} \leq E_{l,K=1}$, in agreement once again with the SUSY approach. However, the next intertwined Hamiltonian $H_{l-2}^{(mk)}$ is derived by adding a new level at $\epsilon_l^{(k)} > \epsilon_l^{(m)}$, *i.e.*, *above* the ground state energy level of $H_{l-1}^{(m)}$, which is against the SUSY *doctrine* but it is valid in our treatment.

A final point is that the equality of the two Hamiltonians $H_{l-2}^{(km)}$ and $H_{l-2}^{(mk)}$, gotten of two iterations of the first order transformations, leads to two different factorizations of the global second order intertwining operator

$$B_{l-2}^{(km)} = a_{l-1}^{(km)} a_l^{(k)} = a_{l-1}^{(mk)} a_l^{(m)} = B_{l-2}^{(mk)}$$

3.4 Particular two-parametric families

The simplest *two*-parametric family of potentials (52) corresponds to $k = -1$ and $m = 0$. In this case, by means of (51) we get the solution to (14):

$$\eta(r) = \frac{1-2l}{l^2(l-1)^2} \left\{ \frac{d}{dr} \ln \left(\frac{e^{r/l(l-1)} \Phi_l^{(-1)}(r)}{\Phi_l^{(0)}(r)} \right) \right\}^{-1}, \quad l > 1, \quad (58)$$

which has been recently reported in [14], where $\Phi_l^{(0)}(r)$, and $\Phi_l^{(-1)}(r)$ are given by (47) and (49), respectively. The appropriate *two*-parametric domain (see (53) and (54)) is given by $\lambda_l^{(-1)}, \lambda_l^{(0)} \in (-\infty, 1)$. From (57) and (34) it is clear that $V_{l-2}^{(-1,0)}(r)$ is strictly isospectral to $V_{l-2}(r)$. Notice that $\lambda_l^{(-1)} = \lambda_l^{(0)} = 0$ leads to $V_{l-2}^{(-1,0)}(r) = V_{l-2}(r)$. Moreover, for $\lambda_l^{(0)} = 0$ and $\lambda_l^{(-1)} = [(l-1)/2]^{2l-1} (2l-2)! / \gamma_{l-1}$, the potentials (52) become the family (46) derived by Fernández, provided l is changed by $l+1$. We make concrete now the discussion of the cases (I) and (II) of Section 3.3.

(I) Departing from the Hamiltonian H_l , by means of a first order intertwining operator we get the Hamiltonian $H_{l-1}^{(0)}$, whose potential is given by (46), where we have added the new level $\epsilon_l^{(0)} = -1/l^2$ below the ground state energy level of H_l , $E_{l,K=1} = -1/(l+1)^2$. The iteration of this procedure leads to $H_{l-2}^{(0,-1)}$, whose potential is given by (52) and (58). In this second step we have added the new level $\epsilon_l^{(-1)} = -1/(l-1)^2$ below the previous ground state energy level $\epsilon_l^{(0)} = -1/l^2$, which agrees with the usual SUSY assumption.

(II) We depart now from H_l to get $H_{l-1}^{(-1)}$, whose potential is given by (48). In this step, we have added the new level $\epsilon_l^{(-1)} = -1/(l-1)^2$ below the ground state energy level of H_l but we have left a *hole* to be filled during the second step (see Section 3.2.2). The next step provides now $H_{l-2}^{(-1,0)}$, whose potential is given again by (52) and (58). Notice that $\beta_{l-1}^{(-1,0)}(r) = \beta_{l-1}^{(0,-1)}(r) + \beta_l^{(0)}(r) - \beta_l^{(-1)}(r)$. The new energy level $\epsilon_l^{(0)} = -1/l^2$ is now added *above* the ground state energy level

$\epsilon_l^{(-1)} = -1/(l-1)^2$ of $H_{l-1}^{(-1)}$ but below the ground state energy level $-1/(l+1)^2$ of H_l . This last procedure fills the *hole* generated during the first step of the whole procedure which is not typical in the usual SUSY assumption.

A different choice of k and m produces families whose spectra are almost equal to the corresponding hydrogen-like spectra, with the two first energy levels different from the levels of the corresponding hydrogen-like potential. For instance, if $k = -3$, $m = 0$, and $l = 4$, the potential $V_2^{(-3,0)}(r)$ has a spectrum of the form $-1/n^2$ but without the levels at $n = 2$ and $n = 3$, similarly as it happens in the first order case mentioned in Section 3.2.3. This is illustrated in Figure 2. A more exotic case is obtained if $k = -4$, $m = -1$, and $l = 5$, where the levels with $n = 2$, $n = 3$, and $n = 5$ are not present for the potential $V_3^{(-4,-1)}(r)$.

4 Concluding remarks

The n th-order intertwining technique developed in this paper is intended to derive n -parametric families of potentials which can be isospectral, or almost isospectral to some well known potential. In particular, we have successfully derived new potentials from the hydrogen-like atom for the one parametric and two parametric cases. We have shown also that the derivation of *multi*-parametric families of almost isospectral potentials becomes mathematically easy by the iteration of the first order intertwining technique. Although the only trouble in our approach could be to find the solutions to a set of n Riccati type equations, characterized by n different factorization energies, we have shown that the eigenvalues problem for the initial potential allows one to get that solutions by means of *unphysical* eigenfunctions. Our *one*-parametric results for the hydrogen-like atom show how the technique effectively works, moreover, our resulting families of potentials are more general than other results previously derived [1, 2, 4, 9]. On the other hand, our *two*-parametric potentials represent a further generalization of the *one*-parametric cases presented in this paper. Finally, we have shown that the construction of a SUSY partner Hamiltonian using a factorization energy ϵ less than the ground-state energy of the departure Hamiltonian is not a general rule in SUSY QM. In particular, the generation of *holes* in the spectra of the n -parametric families of almost isospectral hydrogen-like potentials becomes the key-stone in the construction of the *atypical* n -SUSY partners of the Hydrogen potential derived in this paper. These techniques can be applied to other physically interesting situations, e.g., for systems with continuous spectra as the free particle. A detailed study in this direction will be given elsewhere [27].

Acknowledgements

This work has been supported by CONACyT, Mexico (*Programa de Estancias Posdoctorales en Instituciones del extranjero 1997-1998*). The author would like to thank to Dr. D. J. Fernández for fruitful discussions and suggestions. This work has been benefited from the comments of Dr. L. M. Nieto and has been partially supported by Junta de Castilla

y León, Spain (project C02/97). The kind hospitality at Departamento de Física Teórica and the suggestions of the referees are also acknowledged.

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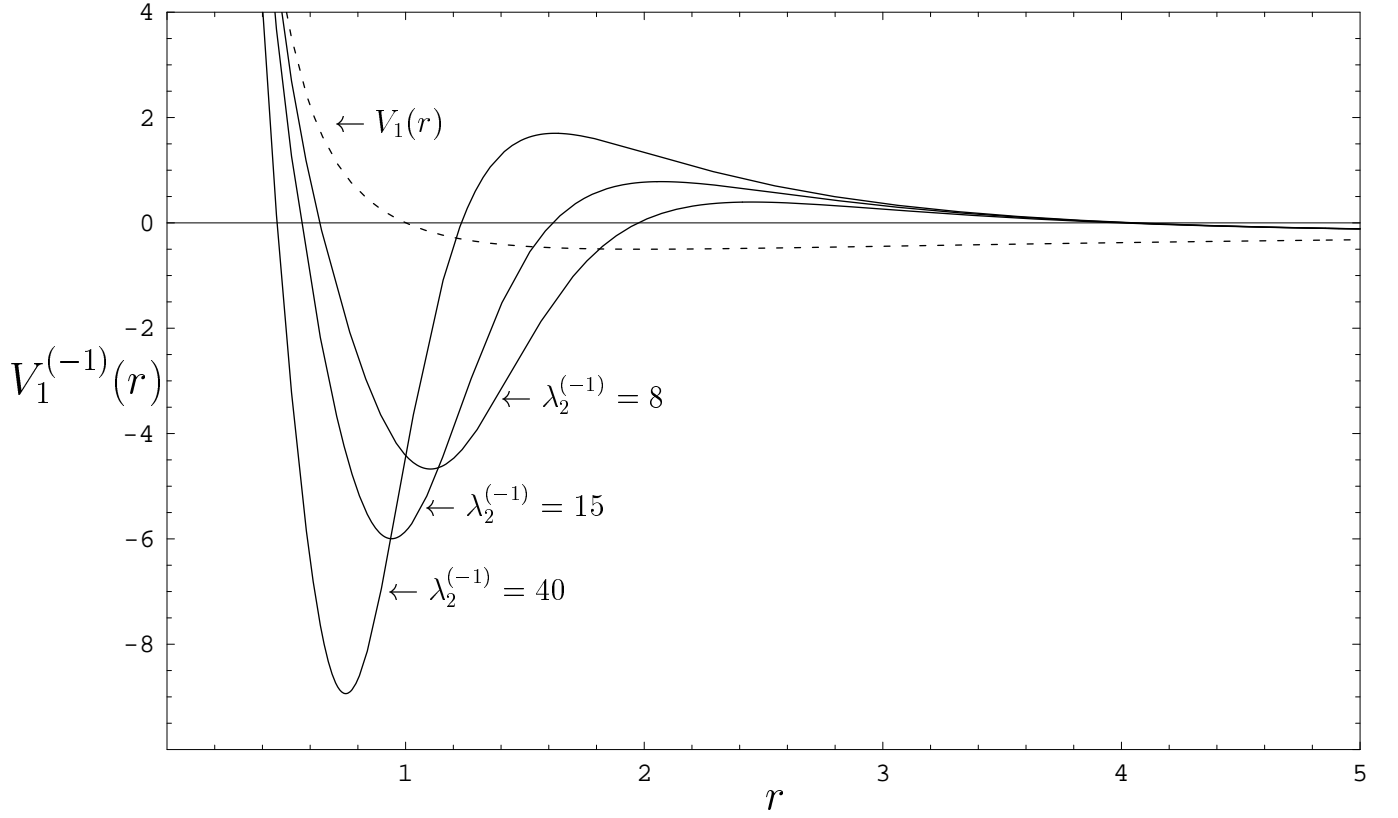


Figure 1: Some members of the *one*-parametric family of potentials (48) for $l = 2$ and the indicated values of $\lambda_2^{(-1)}$. The broken curve represents the hydrogen-like potential $V_{l-1}(r)$. Notice the arising of global minimum for $V_1^{(-1)}(r)$, which produces a well which can be modified by changing the values of the parameter $\lambda_2^{(-1)}$.

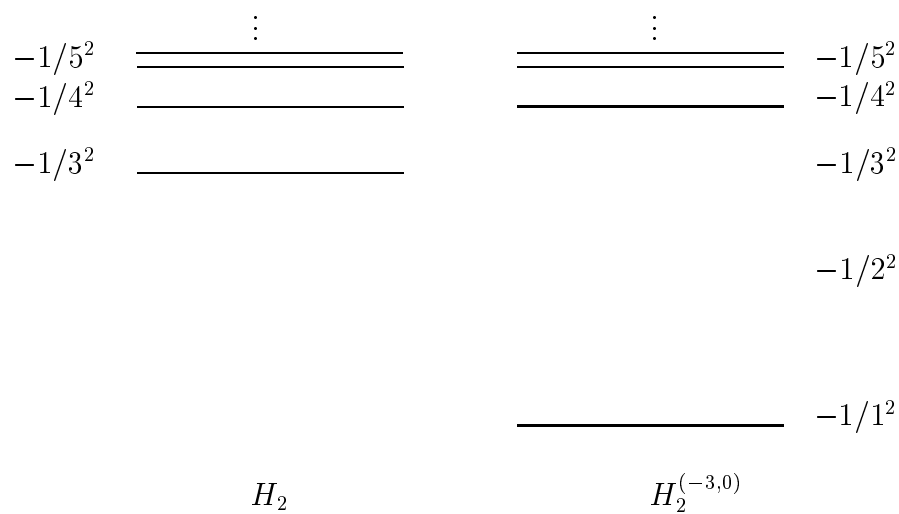


Figure 2: Diagram showing the first energy levels of H_2 and $H_2^{(-3,0)}$. Notice the gap between the ground state energy level $\epsilon_4^{(-3)} = -1$ and the first excited state $\epsilon_4^{(0)} = -1/4^2$ of the *two*-parametric family of Hamiltonians $H_2^{(-3,0)}$.